Mathematics I - Version A - 05/06/2026.

Part I

a) $cl(S) = [2,6] \cup \{7,8,9\}$ ✓

$S' = [2,6]$ ✓

$\max S = 9$ ✓

$x = 3$ ✓ for instance

b) $r = 1/3$ ✓ $g_1 = 18$ ✓

$g_n = 18 \cdot \left(\frac{1}{3}\right)^{n-1}$ ✓

$$\sum_{n=2}^{+\infty} g_n = \frac{g_2}{1-r} = \frac{6}{1-\frac{1}{3}} = 9$$ ✓

c) $D_{g\circ f} =]-\infty, -1[\cup [2, +\infty[$ ✓

$$f'(x) = \frac{(x+1)-(x-2)}{(x+1)^2} = \frac{3}{(x+1)^2}$$ ✓

$g'(x) = \frac{1}{2\sqrt{x}}$ ✓ $\quad f(3) = \frac{1}{4}$ ✓

$$(g\circ f)'(3) = g'(f(3)) \cdot f'(3) = \frac{1}{2\cdot\sqrt{\frac{1}{4}}} \times \frac{3}{4^2} =$$

$$= \boxed{\frac{3}{16}}$$ ✓

$y = 1$ ✓

②

d) $f''(a) = 8$. ✓

upwards ✓

e) $f(3) = 4$ ✓ $f'(3) = -2$ ✓

$f(3{,}05) = f(3) + f'(3)\,(3 - 3{,}05) =$

$= 4 - 2 \cdot (+0.05) = 3{,}9$ ✓

f) $\lim_{x \to +\infty} f(x) = 1$ ✓

$F'(x) = 2x \cdot e^{-x^2}$ ✓

increasing ✓

g) $\bar{u}\,\bar{v} = 0$ ✓ $\|\bar{u}\| = \sqrt{14}$ ✓

$k = +2$ ✓

$2\bar{u} - 3(\bar{v} + \bar{w}) = (-2, -11, 9)$ ✓

h) $\lambda_1 = 2$ $\lambda_2 = 3$ ✓

$\det M^n = 6^n$ ✓

$(M^T)^{-1} = \begin{bmatrix} 1/2 & 0 \\ -1/6 & 1/3 \end{bmatrix}$. ✓

i) consistent and undefined

degree of freedom : 1

$r(A) = 3$

Part II

1) $\sum_{i=1}^{n} (2i-1) = n^2$

$n=1$: $2.1-1 = 1^2$

$1 = 1$ (4)

(6)

Hypothesis $\sum_{i=1}^{n} (2i-1) = n^2$

thesis: $\sum_{i=1}^{n+1} (2i-1) = (n+1)^2$

Proof: $\sum_{i=1}^{n+1} (2i-1) = \sum_{i=1}^{n} (2i-1) + 2(n+1) - 1$

hypothesis

$= n^2 + 2n + 2 - 1 = n^2 + 2n + 1 = (n+1)^2$

4 A

2) a) f is continuous

$\Downarrow$

f is continuous at $x=0$

$\Downarrow$

$$\underbrace{\lim_{x\to 0} \frac{\sqrt{1+kx}-1}{x}}_{\lim_{x\to 0^+} f(x)} = f(1) = 1 \qquad (4)$$

$\Downarrow$

$$\lim_{x\to 0} \frac{1}{2} \cdot \frac{k}{(1+kx)^{1/2}} = 1 \qquad (3)$$

$\Downarrow$

$$\frac{1}{2} k = 1 \quad \Rightarrow \quad \boxed{k=2} \quad ✓ \qquad (3)$$

b). $$(f^+)'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} =$$

$$= \lim_{h\to 0} \frac{\frac{\sqrt{1+2h}-1}{h} - 1}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{1+2h}-1-h}{h^2} = \left(\frac{0}{0}\right)$$

Cauchy

$$\stackrel{\downarrow}{=} \lim_{h\to 0} \frac{\frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{\sqrt{1+2h}} - 1}{2h} =$$

$$= \lim_{h\to 0} \frac{(1+2h)^{-1/2} - 1}{2h}$$

$$= \lim_{h\to 0} \frac{-\frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{(1+2h)^{3/2}}}{2} = -\frac{1}{2} \checkmark \quad (5)$$

$$(f^-)'(0) = \lim_{h\to 0} \frac{e^{ah} + bh - 1}{h} =$$

$$= \lim_{h\to 0} \frac{a e^{h} + b}{1} = a+b.$$

$$\boxed{a+b = -1/2} \checkmark \quad (5)$$

c) $f'(c) \underset{\substack{\uparrow \\ c<0}}{=} a e^{ac} + b \underset{\substack{\uparrow \\ a=2,\, b=1}}{=} 2e^{2c} + 1$

$$f(-1) = e^{-a} - b = e^{-2} - 1 \checkmark$$

$$\frac{f(0) - f(-1)}{1} = \frac{1 - e^{-2} + 1}{1} = 2 - e^{-2} \checkmark \quad (3)$$

4 B

$$2e^{2c}+1 = 2 - e^{-2}$$

$$2e^{2c} = 1 - e^{-2}$$

$$e^{2c} = \frac{1-e^{-2}}{2}$$

$$2c = \ln\left(\frac{1-e^{-2}}{2}\right)$$

$$\boxed{c = \frac{1}{2}\ln\left(\frac{1-e^{-2}}{2}\right)} \checkmark \quad (7)$$

3) a) $u = 1+x^2$ (3)

$du = 2x\,dx \implies \frac{du}{2} = x \cdot dx$

(7)

$$\int \frac{x}{\sqrt{1+x^2}}\,dx = \int \frac{1}{2\sqrt{u}}\,du = \frac{1}{2}\int u^{-1/2}\,du$$

$$= \frac{1}{2}\frac{u^{1/2}}{1/2} + C = (1+x^2)^{1/2} + C \quad , C \in \mathbb{R}$$

$$= \sqrt{1+x^2} + C, \; C \in \mathbb{R}.$$

$\checkmark$

b) $$\int_0^{\sqrt{3}} f(x)\,dx \overset{\text{Barrow's law}}{=} \left[\sqrt{1+x^2}\right]_0^{\sqrt{3}} =$$

(10)

$$= \sqrt{1+3} - 1 = 2-1 = 1$$ ✓

(2)

4a) $$\det M \underset{\text{Laplace, 1st line}}{=} \lambda\,(\lambda^2-1) - 1\,(\lambda-1) + 1\,(1-\lambda)$$

$$= \lambda\,(\lambda^2-1) - 2(\lambda-1)$$ ✓

$$= (\lambda-1)\,(\lambda\,(\lambda+1) - 2)$$ ✓

$$= (\lambda-1)\,(\lambda^2+\lambda-2) = (\lambda-1)\,(\lambda-1)(\lambda+2)$$ ✓

$$= \lambda^3 - 3\lambda + 2$$

$$\det M \neq 0 \overset{(6)}{\Longrightarrow} \lambda \neq 1 \wedge \lambda \neq -2$$ ✓

$\underset{(2)}{\Longrightarrow}$ columns of M are lin. independent ✓

b) $\det P = 2 \neq 0 \Longrightarrow P$ is invertible. (3)

$$\det(2\,P^{-1}\,M^T) =$$

$$= \underset{(2)}{2^3} \cdot \underset{(2)}{\frac{1}{\det P}} \cdot \det M = \frac{8}{2} \cdot (\lambda^3 - 3\lambda + 2) =$$ ✓

$$= 4\,(\lambda^3 - 3\lambda + 2)$$ ✓ (3)

c) $M - M^T = \bar{0}$

$\uparrow$

(10) M is symmetric

$(M - M^T)\bar{v} = \bar{0} \Leftarrow v \in \mathbb{R}^3$ and $\boxed{\lambda \in \mathbb{R}}$ (any)

5

$$\left[\begin{array}{ccc|c} a-1 & 1 & 1 & 1 \\ 1 & a-1 & 1 & b \\ 1 & 1 & a-1 & b^2 \end{array}\right]$$

$\underbrace{\qquad\qquad}_{A}$ $\underbrace{\quad}_{B}$

$a-1=\lambda$

(Compare with ex 4.)

$\det A \neq 0 \iff a \neq 2 \wedge a \neq -1$

(5) $\Rightarrow r(A|B) = r(A) = 3 \Rightarrow$ System is consistent and defined. $\forall b \in \mathbb{R}$

$\boxed{a = 2}$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & b \\ 1 & 1 & 1 & b^2 \end{array}\right]$$

(4) if $\boxed{b = 1} \Rightarrow$ system is consistent and undefined

(4) if $\boxed{b \neq 1} \Rightarrow$ system is unconsistent.

$\boxed{a = -1}$ $\Longrightarrow$ Unconsistent $\forall b$.

(7) $\underbrace{\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}}$ $\left.\begin{matrix} 1 \\ b \\ b^2 \end{matrix}\right]$

$\downarrow$

summing all equations we get

$$0 = 1 + b + b^2 \quad \text{(no solutions!)}$$

+1 Final answer:

- $a \neq 2 \wedge a \neq -1 \Longrightarrow$ Consistent and defined
- $a = 2 \wedge b = 1 \Longrightarrow$ Consistent and undefined
- $a = 2 \wedge b \neq 1 \Longrightarrow$ unconsistent
- $a = -1 \Longrightarrow$ unconsistent